Journal of Statistical Physics, Vol. 119, Nos. 5/6, June 2005 (© 2005)
DOI: 10.1007/s10955-005-3766-x

Nonlinear Double Well Schrödinger Equations in the Semiclassical Limit

Andrea Sacchetti

Received September 28, 2004; accepted January 21, 2005

We consider time-dependent Schrödinger equations with a double well potential and an external nonlinear, both local and non-local, perturbation. In the semiclassical limit, the finite dimensional eigenspace associated to the lowest eigenvalues of the linear operator is almost invariant for times of the order of the *beating period* and the dominant term of the wavefunction is given by means of the solutions of a finite dimensional dynamical system. In the case of local nonlinear perturbation, we assume the spatial dimension $d=1$ or $d=2$.

KEY WORDS: Nonlinear Schrödinger operator; Gross-Pitaevskii equation; Norm estimate of solutions.

1. INTRODUCTION

The theoretical analysis of time-dependent nonlinear Schrödinger (hereafter NLS) equations

$$\begin{cases} i\hbar\dot{\psi} = H_0\psi + \epsilon W\psi, & \epsilon \in \mathbb{R}, \quad \dot{\psi} = \frac{\partial \psi}{\partial t}, \\ \psi(x,0) = \psi^0(x) \in L^2(\mathbb{R}^d), & \|\psi^0\| = 1, \end{cases} \tag{1}$$

where

$$H_0 = -\frac{\hbar^2}{2m}\Delta + V, \quad \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}, \quad d \geqslant 1, \ \ 2m = 1 \tag{2}$$

Dipartimento di Matematica, Università di Modena e Reggio Emilia, Via Campi 213/B, I-41100 Modena, Italy; e-mail: Sacchetti@unimo.it

is the linear Hamiltonian and where $W = W(x, |\psi|)$ is a nonlinear perturbation, has attracted an increasing interest in these last few years (see, e.g., refs. 12 and 15).

In this paper, we consider the case of symmetric potentials V with *double-well* shape. In fact, such a potential appears in several fields, from Bose–Einstein condensate states weakly coupled[13] to localization of symmetric molecules.[11] If the nonlinear term is absent then the linear Hamiltonian H_0 has even-parity and odd-parity eigenstates and the state ψ generically performs a *beating motion*, hence the *beating period* plays the role of unit of time. When we restore the nonlinear term a symmetry breaking phenomenon occurs: that is, if the nonlinearity is larger than a threshold value then new asymmetric stationary states appears.[1,7,8] Furthermore, for higher nonlinearity, the beating motion is forbidden.[8,13,16] These results can be obtained by reducing the NLS equation to a finite dimensional dynamical system exactly solvable and proving the stability of this approximation for times of the order of the beating period with a rigorous estimate of the error in the semiclassical limit.[8,14]

More precisely, in ref. 8 has been considered the case on NLS in any dimension $d \geqslant 1$ where the nonlinear perturbation is *non-local*, that is it is given by

$$W = \langle \psi, g\psi \rangle g(x), \tag{3}$$

where $g(x)$ is a given odd function. In ref. 14 has been considered the case of NLS in dimension $d = 1$ and with a nonlinear *local cubic* perturbation given by

$$W = |\psi|^{2\sigma}, \quad \sigma = 1. \tag{4}$$

Here, we consider NLS Eq. (1) in the semiclassical limit in the cases of both *local* (3) and *non-local* (4), with any $\sigma > 0$, nonlinearity, where we assume that $d = 1$ and $d = 2$ in the case of *local* nonlinear perturbation. Under some generic assumptions on the double-well potential, we give the asymptotic behavior of the solution ψ with a precise estimate of the error. In particular, as general results it follows that new asymmetric stationary states appear and the beating motion, between the two wells of a state initially prepared on the two lowest eigenstates, gradually disappears for increasing nonlinearity.

Hence, the results previously obtained by refs. 8 and 14 can be seen as a particular case of the general treatment given here.

Our paper is organized as follows.

In Section 2, we introduce the assumptions on the potential. Moreover, we collect some semiclassical results concerning the spectrum of the linear Schrödinger operator.

In Section 3, we discuss the beating motion for the unperturbed problem and the choice of the parameters.

In Section 4, we give the existence results for Eq. (1), the conservation laws and *a priori* estimate. The global existence of the solution is proved for both repulsive and attractive nonlinear perturbation, where, in the second case, we have to assume that the strength of the nonlinear perturbation is small enough.

In Section 5, we introduce the two-level approximation and we discuss, in such an approximation, the appearance of new asymmetric stationary states for large nonlinearity strength. The two-level approximation, roughly speaking, consists in projecting Eq. (1) onto the two-dimensional space spanned by the eigenvectors of the linear Schrödinger operator associated to the two lowest eigenvalues. For practical purposes, it is more convenient to choose, as a basis of such a two-dimensional space, the two *single*-*well* states. The dynamical system which we obtain is, in some cases (for instance for cubic local nonlinearity), exactly solvable.

In Section 6, we prove the stability of the two-level approximation in the semiclassical limit. We make use of the comparison criterion between ordinary differential equations and of *a priori* estimate of the solution of the NLS equation.

In Appendix, we recall some useful inequalities.

We close this section by introducing some notations:

– Here $\|\cdot\|_p$ denotes the norm of the Banach space $L^p(\mathbb{R}^d)$, $p \in [1,+\infty]$, $\|\cdot\|$ usually denotes the norm of the space $L^2(\mathbb{R}^d)$ and sometimes (when this does not cause misunderstanding) it denotes the norm of a bounded operator, too;

– The notations $y = o(\hbar^\alpha), y = O(\hbar^\alpha), \alpha \geqslant 0$, and $y = O(e^{-\Gamma/\hbar})$ respectively mean that $y\hbar^{-\alpha} \to 0$ as $\hbar \to 0$ and that there exist $\hbar^\star > 0$ and a positive constant $C > 0$, independent of $\hbar$, such that

$$|y| \leqslant C\hbar^\alpha \quad \text{and} \quad |y| \leqslant Ce^{-\Gamma/\hbar}, \quad \forall \hbar \in (0,\hbar^\star).$$

– The notation $y = \tilde{O}(e^{-\Gamma/\hbar})$ means that for any $\Gamma', 0 < \Gamma' < \Gamma$, then $y = O(e^{-\Gamma'/\hbar})$; that is, there exist $\hbar^\star > 0$ and a positive constant $C = C_{\Gamma'} > 0$, independent of $\hbar$, such that

$$|y| \leqslant C e^{-\Gamma'/\hbar}, \quad \forall \hbar \in (0, \hbar^\star).$$

As usual, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integer numbers and $\mathbb{N}^\star = \mathbb{N} \cup \{0\}$; C denotes any positive constant independent of $\hbar$ and t.

2. ASSUMPTIONS AND PRELIMINARY RESULTS

2.1. Linear Operator

Here, we introduce the assumptions on the double-well potential V and we collect some well known results on the linear operator H_0.

Hypothesis 1. The potential $V(x)$ is a real-valued function such that:

(i) V is a symmetric potential; hence, the Hamiltonian H_0 is invariant under some space inversion $\mathcal{S} : [\mathcal{S}, H_0] = 0$;

(ii) $V \in C^\infty(\mathbb{R}^d)$;

(iii) V(x) admits two minima at $x = x_\pm$, where $x_- = \mathcal{S} x_+$ and $x_+ \neq x_-$, such that

$$V(x) > V_{\min} = V(x_\pm), \quad \forall x \in \mathbb{R}^d, \ \ x \neq x_\pm; \tag{5}$$

(iv) finally we assume that

$$V_\infty^- = \liminf_{|x| \to \infty} V(x) > V_{\min}, \tag{6}$$

and

$$V_\infty^+ = \limsup_{|x| \to \infty} V(x) < +\infty. \tag{7}$$

Remarks.

– For the sake of definiteness we can always assume that, by means of a suitable choice of the coordinates, V is symmetric with respect to the spatial coordinate x_1, that is

$$V(-x_1, x_2, \ldots, x_d) = V(x_1, x_2, \ldots, x_d), \quad \forall (x_1, \ldots, x_d) \in \mathbb{R}^d; \tag{8}$$

– For the sake of simplicity, we assume also that

$$\nabla V(x_\pm) = 0 \quad \text{and} \quad \text{Hess } V(x_\pm) > 0.$$

The case of degenerate minima, that is $\det[\text{Hess } V(x_\pm)] = 0$, could be treated in a similar way; however, we don't dwell here on such details;

– In fact, we could replace assumption (ii) with the weaker assumption $V \in C^2$;

– Assumption (7) is introduced in order to obtain the local existence of the solution of the Cauchy problem (1) by means of well known results[4] (see also refs. 5 and 6). In fact, the local solution of the Cauchy problem exists for some unbounded potentials too (see, e.g., ref. 3 and §9.2 in ref. 5).

The operator H_0 formally defined by (2) admits a self-adjoint realization (still denoted by H_0) on $L^2(\mathbb{R}^d)$ (Theorem III.1.1 in ref. 2).

Let $\sigma(H_0) = \sigma_\text{d} \cup \sigma_\text{ess}$ be the spectrum of the self-adjoint operator H_0, where σ_d denotes the discrete spectrum and σ_ess denotes the essential spectrum. It follows that (see Theorem III.3.1 in ref. 2)

$$\sigma_\text{d} \subset (V_\text{min}, V_\infty^-) \quad \text{and} \quad \sigma_\text{ess} = [V_\infty^-, +\infty).$$

Furthermore, the following two Lemmas hold:

Lemma 1. Let σ_d be the discrete spectrum of H_0. Then, for any $\hbar \in (0, \hbar^\star)$, for some $\hbar^\star > 0$ fixed, it follows that:

(i) σ_d is not empty and, in particular, it contains two eigenvalues at least;

(ii) let $\lambda_\pm$ be the lowest two eigenvalues of H_0, they are non-degenerate, in particular $\lambda_+ < \lambda_-$ and there exists $C > 0$, independent of $\hbar$, such that

$$\inf_{\lambda \in \sigma(H_0) - \{\lambda_+, \lambda_-\}} [\lambda - \lambda_\pm] \geqslant C\hbar. \tag{9}$$

Proof. The proof is an immediate consequence of the above assumptions and standard WKB arguments. In fact, by assuming, for the sake of simplicity, that

$$\text{Hess} V(x_\pm) = 2\,\text{diag}(\mu_1, \mu_2, \ldots, \mu_d), \quad \mu_j > 0, \ \ j = 1, \ldots, d, \tag{10}$$

then the first two eigenvalues of H_0 are given by the semiclassical *single-well* non-degenerate eigenvalues (Theorem 2.3.1 in ref. 10)

$$\lambda_{\pm} = V_{\min} + \left[\sum_{j=1}^{d} \sqrt{\mu_j} \right] \hbar + o(\hbar), \quad \text{as } \hbar \to 0. \tag{11}$$

Furthermore, from conditions (5) and (6) the estimate (9) follows (Corollary 2.3.5 in ref. 10). ∎

Lemma 2. Let $\varphi_{\pm}$ be the normalized eigenvectors associated to $\lambda_{\pm}$, then:

(i) $\varphi_{\pm}$ can be chosen to be real-valued functions such that

$$\varphi_{\pm}(-x_1, x_2, \ldots, x_d) = \pm \varphi_{\pm}(x_1, x_2, \ldots, x_d); \tag{12}$$

(ii) $\varphi_{\pm} \in H^1(\mathbb{R}^d)$;

(iii) $\varphi_{\pm} \in L^p(\mathbb{R}^d)$ for any $p \in [2, +\infty]$;

(iv) there exists a positive constant C, independent on $\hbar$, such that

$$\|\varphi_{\pm}\|_p \leqslant C \hbar^{-d \frac{p-2}{4p}}, \quad \forall p \in [2, +\infty], \quad \forall \hbar \in (0, \hbar^{\star}). \tag{13}$$

Proof. Property (i) immediately follows from (8). Property (ii) follows from Lemma III.3.1 in ref. 2. In order to prove the statement (iii) we recall that the eigenvectors $\varphi_{\pm}$ satisfy to the following global estimate: for any $\delta > 0$ fixed there exists a positive constant $C_{\delta,\hbar} > 0$ such that (Theorem III.3.2 and Corollary III.3.1 in ref. 2)

$$|\varphi_{\pm}(x)| \leqslant C_{\delta,\hbar} \exp[-\delta |x| / \hbar].$$

Hence $\varphi_{\pm} \in L^{\infty}(\mathbb{R}^d)$. From this fact and since $\varphi_{\pm} \in L^2(\mathbb{R}^d)$ then statement (iii) immediately follows. Finally, in order to prove the statement (iv) let

$$\varphi_{\pm} = \frac{1}{\sqrt{2}} [\varphi_R \pm \varphi_L],$$

where the vectors $\varphi_{R,L}$, usually called *single-well states*, are such that

$$\varphi_R = \frac{1}{\sqrt{2}}[\varphi_+ + \varphi_-] \quad \text{and} \quad \varphi_L = \frac{1}{\sqrt{2}}[\varphi_+ - \varphi_-]$$

and

$$\varphi_R(-x_1, x_2, \ldots, x_d) = \varphi_L(x_1, x_2, \ldots, x_d)$$

and they satisfies to the following WKB estimates[10]

$$\varphi_R(x) \sim [2\pi\hbar]^{-d/4} e^{-[\sum_{j=1}^d (x_j - x_{+,j})^2 \sqrt{\mu_j}]/2\hbar}, \quad \text{as } \hbar \to 0, \tag{14}$$

in a neighborhood of the mimina $x_\pm = (x_{\pm,1}, \ldots, x_{\pm,d})$ and where μ_j are defined in (10). From this fact and since $|\varphi_R(x)| \leqslant Ce^{-C/\hbar}$ (see condition (5)) away from the minima, then property (iv) follows for $p = +\infty$. Making use of this estimate, from the normalization of the eigenvectors and from the Hölder inequality then property (iv) follows for any $p \in [2, +\infty]$:

$$\|\varphi_\pm\|_p = \left[\|\varphi_\pm^2 \varphi_\pm^{p-2}\|_1 \right]^{1/p} \leqslant \|\varphi_\pm\|_2^{2/p} \|\varphi_\pm\|_\infty^{(p-2)/p} = \|\varphi_\pm\|_\infty^{(p-2)/p}. \quad \blacksquare$$

From (11) it follows that the *splitting* between the two lowest eigenvalues

$$\omega = \tfrac{1}{2}(\lambda_- - \lambda_+), \tag{15}$$

vanishes as $\hbar$ goes to zero. In order to give a precise estimate of the splitting ω we make use of the fact that V is a symmetric double-well potential with non-zero barrier between the wells. That is, let

$$\Gamma = \inf_\gamma \int_\gamma \sqrt{V(x) - V_{\min}}\, \mathrm{d}x > 0, \tag{16}$$

be the Agmon distance between the two wells; where γ is any path connecting the two wells, that is $\gamma \in AC([0,1], \mathbb{R}^d)$ such that $\gamma(0) = x_-$ and $\gamma(1) = x_+$. From standard WKB arguments (see ref. 10) then it follows that the splitting is *exponentially small*, that is

$$\omega = \tilde{O}(e^{-\Gamma/\hbar}). \tag{17}$$

Furthermore, the *single-well states* $\varphi_{R,L}$ are *localized on one well*, and

$$\|\varphi_R \varphi_L\|_\infty = \tilde{O}(e^{-\Gamma/\hbar}). \tag{18}$$

More precisely, these functions are localized on only one of the two wells in the sense that for any $r > 0$ there exists $C > 0$ such that

$$\int_{D_r(x_+)} |\varphi_R(x)|^2 \, \mathrm{d}x = 1 + O(e^{-C/\hbar})$$

and

$$\int_{D_r(x_-)} |\varphi_L(x)|^2 \, \mathrm{d}x = 1 + O(e^{-C/\hbar}),$$

where $D_r(x_\pm)$ is the ball with center $x_\pm$ and radius r. For such a reason we call them *single-well* (normalized) states.

Remark.

– We emphasize that, by assuming some further regularity properties on the potential V, it is possible to obtain the precise asymptotic behavior of the splitting as $\hbar$ goes to zero.[9]

2.2. Nonlinear Perturbation

Here we admit both *local and non-local* nonlinear perturbations.

Hypothesis 2. Let $g(x)$ be a given real-valued bounded and continuous function. We assume that

(i) **Nonlinear local perturbation.** The perturbation W has the form

$$W = W_\ell(x, |\psi|) = g(x)|\psi(x)|^{2\sigma}, \quad \sigma > 0. \tag{19}$$

(ii) **Nonlinear non-local perturbation.** The perturbation W has the form

$$W = W_{n\ell}(x, |\psi|) = g(x)\langle \psi, g\psi \rangle. \tag{20}$$

In the local perturbation case (19) we assume the dimension $d = 1$ or $d = 2$. In the non-local perturbation case (20) we don't introduce any assumption on the dimension d. Hereafter, let $\tilde{g} = \|g\|_\infty < \infty$.

Remarks.

– If $\sigma = 1$ and $g \equiv 1$ then the NLS Eq. (1) with *local perturbation* (19) coincides with the one previously studied by Sacchetti[14] in dimension $d = 1$. If $g(x)$ is an odd function, that is $g(-x_1, x_2, \dots, x_d) = -g(x_1, x_2, \dots, x_d)$; then the NLS Eq. (1) with *non-local perturbation* (20) coincides with the one previously studied by Grecchi, Martinez and Sacchetti.[8]

– In the case of nonlinear local perturbation (19) we have to assume that the dimension d is not higher than 2. In fact, in the case of dimension $d > 2$ then, provided that $\sigma < \frac{2}{d-2}$, the existence results and the conservation laws (see Section 4) still hold, but the stability result fails (see Section 6).

2.3. Assumption on the Initial State

Let

$$\Pi_c = 1 - [\langle \varphi_+, \cdot \rangle \varphi_+ + \langle \varphi_-, \cdot \rangle \varphi_-]$$

be the projection operator onto the eigenspace orthogonal to the bi-dimensional space associated to the doublet $\{\lambda_\pm\}$. Let ψ^0 be the initial wavefunction, we assume that

Hypothesis 3. $\Pi_c \psi^0 = 0$.

That is, we assume that

$$\psi^0 = c_+ \varphi_+ + c_- \varphi_- = c_R \varphi_R + c_L \varphi_L$$

for some $c_\pm$ and $c_{R,L}$.

Remarks.

– In fact, we could assume that the initial state ψ^0 belongs to a finite dimensional eigenspace of H_0. More precisely, let $\sigma_1, \sigma_2 \subset \sigma(H_0)$ such that $\sigma(H_0) = \sigma_1 \cup \sigma_2, \sigma_1 \cap \sigma_2 = \emptyset, \sigma_1 \subset \sigma_{pp}(H_0)$, where σ_{pp} denotes the pure point spectrum of H_0, and σ_1 has a finite number of elements. Let $\mathcal{H}_1$ be the finite-dimensional spectral eigenspace associated to σ_1. Then we can replace the previous assumption by assuming that $\psi^0 \in \mathcal{H}_1$ and

$$d(\sigma_1, \sigma_2) = \inf_{\lambda \in \sigma_1, \mu \in \sigma_2} |\lambda - \mu| \geqslant C\hbar.$$

In such a case, we have to define $\omega = \frac{1}{2} \inf_{\lambda, \mu \in \sigma_1, \lambda \neq \mu} |\lambda - \mu|$.

3. BEATING MOTION AND CHOICE OF PARAMETERS

Let us consider, for a moment, the time-dependent linear Schrödinger equation

$$\begin{cases} i\hbar\dot{\psi} = H_0\psi, & \dot{\psi} = \frac{\partial\psi}{\partial t}, \\ \psi(x,0) = \psi^0(x) \in L^2(\mathbb{R}^d), & \Pi_c\psi^0 = 0, \end{cases} \tag{21}$$

This equation has an explicit solution given by

$$\begin{aligned} \psi(x,\, t) &= e^{-i\lambda_+ t/\hbar} c_+\varphi_+ + e^{-i\lambda_- t/\hbar} c_-\varphi_- \\ &= \frac{1}{\sqrt{2}}[c_+ e^{-i\lambda_+ t/\hbar} + c_- e^{-i\lambda_- t/\hbar}]\varphi_R + \frac{1}{\sqrt{2}}[c_+ e^{-i\lambda_+ t/\hbar} - c_- e^{-i\lambda_- t/\hbar}]\varphi_L \\ &= \frac{1}{\sqrt{2}} e^{-i\Omega t/\hbar}\left[\left(\tilde{\delta}\varphi_R + \delta\varphi_L\right)\cos(\omega t/\hbar) + i\left(\delta\varphi_R + \tilde{\delta}\varphi_L\right)\sin(\omega t/\hbar)\right], \end{aligned}$$

where we set

$$\lambda_\pm = \Omega \mp \omega, \quad \tilde{\delta} = c_+ + c_-, \quad \delta = c_+ - c_-.$$

That is $\psi(x,t)$ performs a *beating motion* with *beating period*

$$T = \frac{2\pi\hbar}{\omega}.$$

Such a period will play the role of unit of time.

Hypothesis 4. Let ω be the splitting (15) satisfying to the asymptotic estimate (17). We assume that the real-valued parameter ϵ depends on $\hbar$ in such a way

$$\frac{|\epsilon|\hbar^{-d\sigma/2}}{\omega} \leqslant C, \quad \forall\hbar \in (0,\, \hbar^\star) \tag{22}$$

for some positive constant C, independent of $\hbar$, and for some $\hbar^\star$, where σ is defined in (19) for nonlinear local perturbations and where $\sigma = 0$ for nonlinear non-local perturbations (20).

Remarks.

– We emphasize that the strength of the perturbation is, roughly speaking, given by $|\epsilon|$ is the case of nonlinear non-local perturbation (20), and given by $|\epsilon|\hbar^{-d\sigma/2}$ in the case of nonlinear local perturbation (19). The ratio

$$\eta = \frac{\epsilon\hbar^{-d\sigma/2}}{\omega}$$

plays the role of effective nonlinearity parameter. The above assumption implies that $|\eta| \leqslant C$.

– Condition (22) implies that

$$\begin{aligned}&(\text{Beating period}) \times (\text{Perturbation strength})\\ &= T \times |\epsilon|\hbar^{-d\sigma/2} \approx \frac{\hbar}{\omega} \times \omega = \hbar \approx \text{dist}(\sigma(H_0), \lambda_\pm).\end{aligned}$$

Thus, heuristic arguments do not suggest us that the subspace $(1-\Pi_c)L^2$ is almost invariant for times of the order of the beating period. In fact, we will prove that $\|\Pi_c\psi\| = \tilde{O}(e^{-\Gamma/\hbar})$ for any $t\in[0,T]$ for $\hbar$ small enough.

4. EXISTENCE RESULTS AND CONSERVATION LAWS

Here, making use of some results by ref. 4 (see also ref. 5), we prove that the solution of Eq. (1) globally exists. To this end we recall that $\psi^0 \in H^1 \cap L^p$ for any $p\in[2,+\infty]$ (see Lemma 2). The assumptions on the strength of the nonlinear perturbation, that is $\epsilon = \tilde{O}(e^{-\Gamma/\hbar})$ (see Eqs. (17) and (22)), could, in order to prove the global existence result and the conservation laws, be relaxed; in fact, here we simply require that $\epsilon = O(\hbar^\alpha)$ for some $\alpha > 2$.

4.1. Local Existence

Theorem 1. There exists $T^\star > 0$ and an unique solution $\psi \in C([0, T^\star), H^1) \cap C^1([0, T^\star), H^{-1})$ of (1), where $T^\star = +\infty$ or $\|\nabla\psi\| \to +\infty$ as $t \to T^\star$.

Proof. This result is a consequence of Theorem 2.1 and Examples 1–3 by ref. 4 (see also ref. 5). In fact, $V \in L^\infty$ and $\psi^0 \in H^1$; furthermore we show that both W_l and W_{nl} satisfy the conditions of ref. 4 (see also ref. 5). To this end let, in the case of local perturbation,

$$f(x,\, z) = g(x)|z|^{2\sigma} z,$$

where from (A1) it follows that

$$|f(x, z_1) - f(x, z_2)| \leqslant C[1 + |z_1|^{2\sigma} + |z_2|^{2\sigma}]|z_1 - z_2| \tag{23}$$

for some positive constant C. Then the local existence of the solution in the case of local nonlinear perturbation follows. In the non-local case, where

$$W = W_{n\ell}(x, \psi) = g(x)\langle \psi, g\psi \rangle$$

the local existence result follows by means the same arguments. We simply have to check that

$$\| W_{n\ell}(x, u)u - W_{n\ell}(x, v)v \| \leqslant C \| u - v \|.$$

Indeed,

$$\begin{aligned}
&\| W_{n\ell}(x, u)u - W_{n\ell}(x, v)v \| \\
&\quad = \tilde{g} \| \langle u, gu \rangle u - \langle v, gv \rangle v \| \\
&\quad = \tilde{g} \| (\langle u, gu \rangle - \langle v, gv \rangle) u + \langle v, gv \rangle (u - v) \| \\
&\quad \leqslant \tilde{g} \| u \| \; |\langle u, gu \rangle - \langle v, gv \rangle| + \tilde{g}^2 \| v \|^2 \| u - v \| \\
&\quad \leqslant \tilde{g} \| u \| \; |\langle u - v, gu \rangle + \langle v, g(u - v) \rangle| + \tilde{g}^2 \| v \|^2 \| u - v \| \\
&\quad \leqslant \tilde{g}^2 \| u \| \; [\| u \| \cdot \| u - v \| + \| v \| \cdot \| u - v \|] + \tilde{g}^2 \| v \|^2 \| u - v \| \\
&\quad \leqslant C \| u - v \|,
\end{aligned}$$

where

$$C = \tilde{g}^2 [\| u \|^2 + \| v \|^2 + \| u \| \cdot \| v \|]. \quad \blacksquare$$

4.2. Conservation Laws

By means of a direct computation the following first integral exists.

4.2.1. Conservation of the norm

Let

$$\mathcal{N}(\psi) = \|\psi\|^2$$

then

$$\mathcal{N}[\psi(x,t)] = \mathcal{N}[\psi^0(x)] = 1.$$

4.2.2. Conservation of the energy

Let us consider the case of local nonlinear perturbation (19). Let

$$\mathcal{H}(\psi) = \mathcal{H}_\ell(\psi) = \langle \psi, H_0\psi\rangle + \frac{\epsilon}{\sigma+1}\langle \psi^{\sigma+1}, g\psi^{\sigma+1}\rangle \tag{24}$$

defined on $H^1(\mathbb{R}^d) \cap L^{2(\sigma+1)}(\mathbb{R}^d)$. Then a direct computation gives that

$$\mathcal{H}_\ell[\psi(x,\, t)] = \mathcal{H}_\ell[\psi^0(x)].$$

Similarly, in the case of non-local nonlinear perturbation (20) then it follows that

$$\mathcal{H}_{n\ell}[\psi(x,t)] = \mathcal{H}_{n\ell}[\psi^0(x)],$$

where the energy is defined as

$$\mathcal{H}(\psi) = \mathcal{H}_{nl}(\psi) = \langle \psi, H_0\psi\rangle + \frac{1}{2}\epsilon\langle \psi, g\psi\rangle^2 \tag{25}$$

on $H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

4.3. A priori Estimates

Theorem 2. Let

$$\Lambda = \frac{\mathcal{H}[\psi^0] - V_{\min}}{\hbar^2},$$

where $\mathcal{H}$ is the energy defined above in Eqs. (24) and (25) for, respectively, local and non-local perturbations. The solution $\psi(x,t)$ of Eq. (1) satisfies to the following a priori estimates

$$\|\nabla\psi\| \leqslant C\sqrt{\Lambda} \tag{26}$$

and

$$\|\psi\|_p \leqslant C\Lambda^{d\frac{p-2}{4p}}, \tag{27}$$

where

$$p\in[2,+\infty] \text{ if } d=1 \quad \text{and} \quad p\in[2,+\infty) \text{ if } d=2 \tag{28}$$

Proof. We consider, at first, the case of local perturbation (19). The conservation of the energy $\mathcal{H}_l(\psi)$ gives that

$$\begin{aligned}\hbar^2\|\nabla\psi\|^2 &= \mathcal{H}_l(\psi^0) - \frac{\epsilon}{\sigma+1}\langle g\psi^{\sigma+1}, \psi^{\sigma+1}\rangle - \langle V\psi, \psi\rangle \\ &\leqslant \mathcal{H}_l(\psi^0) - V_{\min}\mathcal{N}(\psi^0) + \frac{\tilde{g}|\epsilon|}{\sigma+1}\|\psi\|^{2(\sigma+1)}_{2(\sigma+1)},\end{aligned}$$

where

$$V_{\min} = \min_x V(x) = V(x_\pm) > -\infty \quad \text{and} \quad \tilde{g} = \|g\|_\infty.$$

Hence

$$\|\nabla\psi\|^2 \leqslant \Lambda + \rho^2\|\psi\|^{2(\sigma+1)}_{2(\sigma+1)},$$

where

$$\rho^2 = \frac{\tilde{g}|\epsilon|}{(\sigma+1)\hbar^2} \leqslant C|\epsilon|\hbar^{-2} \ll 1 \quad \text{and} \quad \hbar|\Lambda| = C + o(1)$$

since the small parameter ϵ satisfies (22). Now, we make use of the Gagliardo–Nirenberg inequality (A2) obtaining

$$\|\nabla\psi\|^2 \leqslant \Lambda + C\rho^2\|\nabla\psi\|^{\sigma d}\|\psi\|^{2+\sigma(2-d)} \leqslant \Lambda + C\rho^2\|\nabla\psi\|^{\sigma d} \tag{29}$$

from which and from *bootstrap* argument the estimate (26) follows. Indeed, if $\sigma d \leqslant 2$ then the result immediately follows. If $\sigma d > 2$ we recall that φ_R and φ_L satisfy to the asymptotic behavior (14) then $\|\nabla\psi^0\| \leqslant C\hbar^{-1/2}$. If we set

$$y = \hbar^{1/2}\|\nabla\psi\|, \quad \alpha = \hbar\Lambda, \quad \beta = C\rho^2\hbar^{1-\sigma d/2},$$

where initially $y \leqslant C$, and where

$$C^{-1} \leqslant \alpha \leqslant C \quad \text{and} \quad \beta = o(1) \quad \text{as } \hbar \to 0$$

then (29) can be written as

$$y^2 \leqslant \alpha + \beta y^{\sigma d}.$$

Then fixed point arguments proved that $y \leqslant C$ for any time, from which (26) follows. From this fact, and making use of the Gagliardo–Nirenberg inequality again, we obtain that

$$\|\psi\|_p \leqslant C\|\nabla\psi\|^{\delta}\|\psi\|^{1-\delta} \leqslant C\Lambda^{\frac{1}{2}\delta}, \quad \delta = \frac{d(p-2)}{2p}. \tag{30}$$

Now, we consider the case of non-local perturbation (20). In such a case, the conservation of the energy $\mathcal{H}_{nl}$ gives inequality (26) immediately. Indeed,

$$\begin{aligned} \|\nabla\psi\| &= \frac{1}{k^2}\left[\mathcal{H}_{nl}(\psi^0) - \langle V\psi, \psi\rangle - \frac{1}{2}\epsilon\langle\psi, g\psi\rangle^2\right] \\ &\leqslant \Lambda + \rho^2[\|\psi\|_2^2]^2, \quad \rho^2 = \frac{1}{2}\tilde{g}|\epsilon| \ll 1 \\ &\leqslant \Lambda + C\rho^2\|\nabla\psi\| \end{aligned}$$

As above, the estimates $\|\nabla\psi\| \leqslant C\sqrt{\Lambda}$ and (27) follow. ∎

Remarks.

– Since ψ^0 is prepared on the first two states and since (22) then it follows that $\Lambda \sim \hbar^{-1}$. Hence the above estimates take the form

$$\|\nabla\psi\| \leqslant C\hbar^{-1/2} \quad \text{and} \quad \|\psi\|_p \leqslant C\hbar^{-d\frac{p-2}{4p}} \tag{31}$$

for any p satisfying(28);

– In fact the above *a priori* estimates hold for any $\epsilon > 0$ (repulsive nonlinearity), and for any $\epsilon < 0$ (attractive nonlinearity) such that $\epsilon = O(\hbar^{\alpha})$ for any $\alpha > 2$.

4.4. Global Existence

Theorem 3. The solution ψ of (1) globally exists; that is $T^* = +\infty$.

Proof. The global solution immediately follows from Theorem 1 and from the estimate (25). ∎

5. TWO-LEVEL APPROXIMATION

5.1. Two-level Approximation

Since the beating period $T = \frac{2\pi\hbar}{\omega}$ plays the role of the unit of time it is more convenient to consider the *slow time*

$$\tau = \frac{\omega t}{\hbar}.$$

Therefore, if we consider the change of variable (with abuse of notation)

$$\psi(x,t) \to \psi(x,\tau) = e^{i\Omega t/\hbar}\psi(x,t), \quad \Omega = \frac{1}{2}[\lambda_+ + \lambda_-]$$

then Eq. (1) takes the form (here $'$ denotes the derivative with respect to τ)

$$\begin{cases} i\psi' = \frac{1}{\omega}[H_0 - \Omega]\psi + \frac{\epsilon}{\omega}W\psi \\ \psi(x,0) = \psi^0(x) \in L^2(\mathbb{R}^d), \quad \Pi_c\psi^0 = 0, \end{cases} \tag{32}$$

since $W = W(x, |\psi|)$. In fact, we have also replaced the original Hamiltonian H_0 by $H_0 - \Omega$, by means of the gauge choice $e^{i\Omega t/\hbar}$, in order to have a simpler expression of the two-level system (see Eq. (36) below). In order to study this equation for $\tau \in [0, \tau']$, for any fixed $\tau' > 0$, in the semiclassical limit we rewrite ψ in the following form

$$\psi(x,\tau) = \varphi(x,\tau) + \psi_c(x,\tau), \quad \varphi(x,\tau) = a_R(\tau)\varphi_R(x) + a_L(\tau)\varphi_L(x), \tag{33}$$

where

$$\psi_c(x,\tau) = \Pi_c\psi(x,\tau)$$

and where

$$a_R(\tau) = \langle\varphi_R, \psi\rangle \quad \text{and} \quad a_L(\tau) = \langle\varphi_L, \psi\rangle$$

are unknown complex-valued functions. Since

$$\begin{aligned} H_0\psi &= a_R H_0\varphi_R + a_L H_0\varphi_L + H_0\psi_c \\ &= a_R[\Omega\varphi_R - \omega\varphi_L] + a_L[-\omega\varphi_R + \Omega\varphi_L] + H_0\psi_c \end{aligned}$$

then, by substituting (33) in (32) and projecting the resulting equation onto the one-dimensional spaces spanned by the *single-well* states φ_R and φ_L, and on the space $\Pi_c L^2(\mathbb{R}^d)$ it follows that it takes the form

$$\begin{cases} i a_R' = -a_L + r_R & r_R = r_R(a_R, a_L, \psi_c) = \frac{\epsilon}{\omega}\langle \varphi_R, W\psi\rangle \\ i a_L' = -a_R + r_L & r_L = r_L(a_R, a_L, \psi_c) = \frac{\epsilon}{\omega}\langle \varphi_L, W\psi\rangle \\ i\psi_c' = \frac{1}{\omega}[H_0 - \Omega]\psi_c + r_c & r_c = r_c(a_R, a_L, \psi_c) = \frac{\epsilon}{\omega}\Pi_c W\psi\rangle \end{cases} \tag{34}$$

with initial conditions

$$a_{R,L}(0) = \langle \varphi_{R,L}, \psi^0\rangle, \quad \psi_c^0 = \psi_c(x, 0) = \Pi_c\psi^0 = 0.$$

Lemma 3. Let

$$r_{R,L} = r_{R,L}(a_R, a_L, \psi_c) = \frac{\epsilon}{\omega}\langle \varphi_{R,L}, W\psi\rangle.$$

Then, it follows that

$$r_{R,L}(a_R, a_L, 0) = \eta\tilde{r}_{R,L}(a_R, a_L) + \tilde{O}(e^{-\Gamma/\hbar}), \tag{35}$$

where

(i) **Local perturbation:**

$$\tilde{r}_R = C_R|a_R|^{2\sigma}a_R, \quad \tilde{r}_L = C_L|a_L|^{2\sigma}a_L \quad \text{and} \quad \eta = \frac{\epsilon}{\omega}\hbar^{-d\sigma/2},$$

where

$$C_R = \hbar^{d\sigma/2}\langle \varphi_R, g|\varphi_R|^{2\sigma}\varphi_R\rangle \quad \text{and} \quad C_L = \hbar^{d\sigma/2}\langle \varphi_L, g|\varphi_L|^{2\sigma}\varphi_L\rangle$$

are such that $C_{R,L} = O(1)$ as $\hbar$ goes to zero.

(ii) **Non-Local perturbation:**

$$\tilde{r}_R = C_R a_R|a_R|^2 + \tilde{C}a_R|a_L|^2, \quad \tilde{r}_L = C_L a_L|a_L|^2 + \tilde{C}a_L|a_R|^2$$

and

$$\eta = \frac{\epsilon}{\omega},$$

where

$$C_R = \langle \varphi_R, g\varphi_R \rangle^2, \quad C_L = \langle \varphi_L, g\varphi_L \rangle^2 \quad \text{and} \quad \tilde{C} = \sqrt{C_R C_L}$$

are such that $C_{R,L} = O(1)$ as $\hbar$ goes to zero.

Proof. In order to give the explicit expression of the terms $r_{R,L}$ we consider the local and non-local perturbations separately.

Local perturbation. In such a case, we get by taking $\psi_c = 0$ in W_l

$$\begin{aligned} r_R(a_R, a_L, 0) &= \frac{\epsilon}{\omega} \langle \varphi_R, g|a_R\varphi_R + a_L\varphi_L|^{2\sigma}(a_R\varphi_R + a_L\varphi_L)\rangle \\ &= \frac{\epsilon}{\omega}\left[|a_R|^{2\sigma} a_R \langle \varphi_R, g|\varphi_R|^{2\sigma}\varphi_R\rangle + \tilde{O}(e^{-\Gamma/\hbar})\right] \end{aligned}$$

according to (18). Similarly, we obtain that

$$r_L(a_R, a_L, 0) = \frac{\epsilon}{\omega}\left[|a_L|^{2\sigma} a_L \langle \varphi_L, g|\varphi_L|^{2\sigma}\varphi_L\rangle + \tilde{O}(e^{-\Gamma/\hbar})\right].$$

If we set

$$C_R = \hbar^{d\sigma/2}\langle \varphi_R, g|\varphi_R|^{2\sigma}\varphi_R\rangle \quad \text{and} \quad C_L = \hbar^{d\sigma/2}\langle \varphi_L, g|\varphi_L|^{2\sigma}\varphi_L\rangle$$

then

$$r_{R,L} = \eta C_{R,L}|a_{R,L}|^{2\sigma} a_{R,L} + \tilde{O}(e^{-\Gamma/\hbar})$$

for any τ. Furthermore, from Lemma 2 it follows that

$$\begin{aligned} |C_{R,L}| &\leqslant \tilde{g}\hbar^{d\sigma/2}\|\varphi_{R,L}^{\sigma+1}\|^2 = \tilde{g}\hbar^{d\sigma/2}\|\varphi_{R,L}\|_{2(\sigma+1)}^{2(\sigma+1)} \\ &\leqslant C\hbar^{d\sigma/2}\left[\hbar^{-d\frac{2\sigma}{8(\sigma+1)}}\right]^{2(\sigma+1)} \leqslant C, \quad \forall \hbar \in (0, \hbar^\star), \end{aligned}$$

where $\tilde{g} = \|g\|_\infty < +\infty$.

Non-Local perturbation. In such a case, it follows that (where we set $\psi_c = 0$ inside W_{nl})

$$
\begin{aligned}
& r_R(a_R, a_L, 0) \\
&\quad = \frac{\epsilon}{\omega}\langle \varphi_R, (a_R\varphi_R + a_L\varphi_L)g\rangle \cdot \langle (a_R\varphi_R + a_L\varphi_L), g(a_R\varphi_R + a_L\varphi_L)\rangle \\
&\quad = \frac{\epsilon}{\omega}\left[a_R\langle\varphi_R, g\varphi_R\rangle + \tilde{O}(e^{-\Gamma/\hbar})\right] \\
&\qquad \times \left[|a_R|^2\langle\varphi_R, g\varphi_R\rangle + |a_L|^2\langle\varphi_L, g\varphi_L\rangle + \tilde{O}(e^{-\Gamma/\hbar})\right]
\end{aligned}
$$

since (18). If we set

$$
C_R = \langle\varphi_R, g\varphi_R\rangle^2, \quad C_L = \langle\varphi_L, g\varphi_L\rangle^2, \quad \tilde{C} = \sqrt{C_R C_L}
$$

then

$$
\begin{aligned}
r_R &= \eta C_R a_R|a_R|^2 + \eta\tilde{C}a_R|a_L|^2 + \frac{\epsilon}{\omega}\tilde{O}(e^{-\Gamma/\hbar}) \\
&= \eta C_R a_R|a_R|^2 + \eta\tilde{C}a_R|a_L|^2 + \tilde{O}(e^{-\Gamma/\hbar})
\end{aligned}
$$

since (22) and, similarly

$$
\begin{aligned}
r_L &= \eta C_L a_L|a_L|^2 + \eta\tilde{C}a_L|a_R|^2 + \frac{\epsilon}{\omega}\tilde{O}(e^{-\Gamma/\hbar}) \\
&= \eta C_L a_L|a_L|^2 + \eta\tilde{C}a_L|a_R|^2 + \tilde{O}(e^{-\Gamma/\hbar}).
\end{aligned}
$$

Finally

$$
|C_{R,L}| \leqslant \tilde{g}\|\varphi_{R,L}\|^4 \leqslant C. \quad \blacksquare
$$

Definition 1. We call two-level approximation the system of differential equations given by

$$
\begin{cases} ib_R' = -b_L + \eta\tilde{r}_R(b_R, b_L) \\ ib_L' = -b_R + \eta\tilde{r}_L(b_R, b_L), \end{cases} \quad b_{R,L}(0) = a_{R,L}(0). \tag{36}
$$

Remarks.

– The *two-level approximation* is obtained, up to an exponentially small term, by substituting $\psi_c \equiv 0$ inside Eq. (34).

– The solution of Eq. (36) globally exists.

5.2. First Integrals

As for the complete Eq. (1) a direct computation proves the following conservation laws.

5.2.1. Conservation of the Norm

Let

$$\tilde{\mathcal{N}}(b_R, b_L) = |b_R|^2 + |b_L|^2,$$

then

$$\tilde{\mathcal{N}}[b_R(\tau), b_L(\tau)] = \tilde{\mathcal{N}}[b_R(0), b_L(0)], \quad \forall \tau \in \mathbb{R}.$$

In particular $\tilde{\mathcal{N}}(b_R, b_L) = 1$ since

$$\tilde{\mathcal{N}}(b_R, b_L) \equiv |b_R(0)|^2 + |b_L(0)|^2 = |a_R(0)|^2 + |a_L(0)|^2 = \|\psi^0\|^2 = 1. \quad (37)$$

5.2.2. Conservation of the Energy

Let, in the case of local perturbation (19),

$$\begin{aligned}\tilde{\mathcal{H}}(b_R, b_L) &= \tilde{\mathcal{H}}_l(b_R, b_L) \\ &= -\left[(\bar{b}_R b_L + \bar{b}_L b_R) - \frac{\eta}{\sigma+1}\left(|b_R|^{2(\sigma+1)} C_R + |b_L|^{2(\sigma+1)} C_L\right)\right],\end{aligned}$$

or, in the case of non-local perturbation (20),

$$\begin{aligned}\tilde{\mathcal{H}}(b_R, b_L) &= \tilde{\mathcal{H}}_{nl}(b_R, b_L) \\ &= -\left[(\bar{b}_R b_L + \bar{b}_L b_R) - \frac{\eta}{2}\left(|b_R|^4 C_R + |b_L|^4 C_L + 2\tilde{C}|b_R|^2 \cdot |b_L|^2\right)\right].\end{aligned}$$

Then a direct computation gives that

$$\tilde{\mathcal{H}}[b_R(\tau), b_L(\tau)] = \tilde{\mathcal{H}}[b_R(0), b_L(0)], \quad \forall_\tau \in \mathbb{R}.$$

Remarks.

– We emphasize that the two-level system (36) takes the Hamiltonian form

$$i B' = \partial_{\bar{B}} \tilde{\mathcal{H}}, \quad B = (b_R, b_L).$$

5.3. Analysis of the Two-level Approximation

Here, we perform the qualitative analysis of the two-level approximation where, for the sake of simplicity, we assume that the function $g(x)$ is an even function (resp. odd function) for local perturbations (19) (resp. non-local perturbations (20)). We prove that

Theorem 4. There exists a threshold value $\eta^\star > 0$ such that the two-level system (36) admits just two stationary symmetric (that is $|b_R|^2 = |b_L|^2 = \frac{1}{2}$) solutions for any $\eta \in [0, \eta^\star]$. At $\eta = \eta^\star$ a bifurcation phenomenon occurs and for $\eta > \eta^\star$ new stationary asymmetric (that is $|b_{R,L}|^2 \neq \frac{1}{2}$) solutions appear.

Proof. In order to prove this result, we set

$$b_R = pe^{i\alpha}, \quad b_L = qe^{i\beta}, \quad z = p^2 - q^2, \quad \theta = \alpha - \beta,$$

where p and q are such that $p^2 + q^2 = 1$. The *imbalance function* z takes value in the interval $[-1, 1]$; when $z = 1$ then $|b_R| = 1$ and $|b_L| = 0$ and the wavefunction $\varphi = b_R\varphi_R + b_L\varphi_L$ is practically localized on the *right-side* well, in contrast, when $z = -1$ then $|b_R| = 0$ and $|b_L| = 1$ and the wavefunction φ is practically localized on the *left-side* well.

For the sake of definiteness we just consider the local perturbation case, the non-local case could be similarly treated. In such a case, the *two-level approximation* (36) takes the form

$$\begin{cases} z' = 2\sqrt{1-z^2}\sin\theta \\ \theta' = \frac{-2z}{\sqrt{1-z^2}}\cos\theta - \eta\left[C_R\left(\frac{1+z}{2}\right)^\sigma - C_L\left(\frac{1-z}{2}\right)^\sigma\right]. \end{cases} \tag{38}$$

If we set now

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}(z,\theta) = 2\left\{\sqrt{1-z^2}\cos\theta - \frac{\eta}{\sigma+1}\left[C_R\left(\frac{1+z}{2}\right)^{\sigma+1} + C_L\left(\frac{1-z}{2}\right)^{\sigma+1}\right]\right\},$$

then $\tilde{\mathcal{H}}$ is an integral of motion and the above equation takes the Hamiltonian form

$$\begin{cases} \theta' = \partial_z\tilde{\mathcal{H}} \\ z' = -\partial_\theta\tilde{\mathcal{H}}. \end{cases}$$

Since g is an even function then $C_L = C_R$ and it is not hard to see that when the nonlinear perturbation is small enough then the above dynamical system has only two stationary solutions (θ_1, z_1) and (θ_2, z_2) where $\theta_1 = 0$ and $\theta_2 = \pi$ and where $z_1 = z_2 = 0$ is the unique solution of the equation

$$\mp \frac{2z}{\sqrt{1-z^2}} - \eta \left[C_R \left(\frac{1+z}{2} \right)^\sigma - C_L \left(\frac{1-z}{2} \right)^\sigma \right] = 0. \tag{39}$$

In contrast, when the strength of the nonlinear perturbation is larger than a threshold parameter $\eta^\star$ then new solutions $z \neq 0$ of Eq. (39) appear. ∎

In particular:

5.3.1. Cubic ($\sigma = 1$) and Quintic ($\sigma = 2$) Local Nonlinearity

In these cases, Eq. (39) takes the form (we assume $C_R = C_L = 1$ for the sake of definiteness)

$$\frac{2z}{\sqrt{1-z^2}} - \eta z = 0.$$

For η larger than the threshold value $\eta^\star = 2$ then the solution (z_2, θ_2) bifurcates (see Fig. 1(a)) and two new solutions (θ_3, z_3) and (θ_4, z_4) appear, where $\theta_{3,4} = \pi$ and

$$z_{3,4} = \pm \frac{\sqrt{\eta^2 - 4}}{\eta}.$$

5.3.2. Higher Local Nonlinearity ($\sigma = 3$ and $\sigma = 4$)

In the case $\sigma = 3$ then we have a picture similar as before. That is, for η larger than the threshold value $\eta^\star = \frac{8}{3}$ then the solution (z_2, θ_2) bifurcates (see Fig. 1(a)) and two new solutions (θ_3, z_3) and (θ_4, z_4) appear, where $\theta_{3,4} = \pi$ and

$$z_{3,4} = \pm \left[\frac{2\mathcal{X}}{3\eta} + \frac{8\eta}{3\mathcal{X}} - \frac{5}{3} \right]^{1/2}, \quad \eta^\star \leqslant \eta,$$

where

$$\mathcal{X} = \left[\left(8\eta^2 - 108 + 12\sqrt{81 - 12\eta^2} \right) \eta \right]^{1/3}.$$

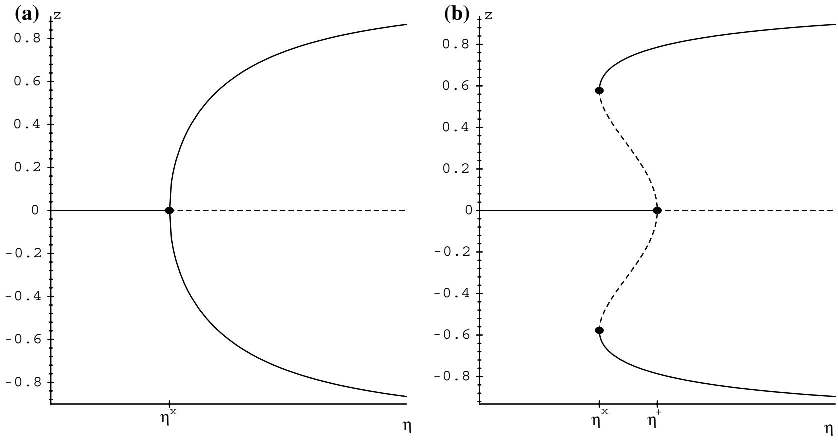

Fig. 1. In this figure, we plot the graph of the solutions (full lines represent stable centers, broken lines represent unstable centers) of Eq. (39) as function of the parameter η, for given nonlinearity. In (a) we consider the cases of $\sigma = 1, 2, 3$, where a pitch-fork bifurcation occurs at $\eta = \eta^\star$, where $\eta^\star = 2$ for $\sigma = 1$, 2 and $\eta^\star = \frac{8}{3}$ for $\sigma = 3$. In (b) we consider the case $\sigma = 4$, in such a case new solutions appear at $\eta = \eta^\star$ where $\eta^\star = \sqrt{27/2}$; at $\eta = \eta^+ = 4$ two of them collapses and, then disappear.

In the case $\sigma = 4$, we have a different picture, that is at η coinciding with the threshold value $\eta^\star = \sqrt{\frac{27}{2}}$ then four new solutions (θ_j, z_j), $j = 3, 4, 5, 6$ appear, where $\theta_j = \pi$ and $z_{3,5} = \frac{1}{\sqrt{3}}$ and $z_{4,6} = -\frac{1}{\sqrt{3}}$. In particular,

$$
\begin{aligned}
z_{3,4} &= \pm \left[\frac{2\mathcal{X}}{3\eta} + \frac{2\eta}{3\mathcal{X}} - \frac{1}{3} \right]^{1/2}, \quad \eta^\star \leqslant \eta, \\
z_{5,6} &= \pm \left[\left(-\frac{\mathcal{X}}{3\eta} - \frac{\eta}{3\mathcal{X}} - \frac{1}{3} \right) - i \frac{\sqrt{3}}{2} \left(\frac{2\mathcal{X}}{3\eta} - \frac{2\eta}{3\mathcal{X}} \right) \right]^{1/2}, \quad \eta^\star \leqslant \eta \leqslant \eta^+ = 4,
\end{aligned}
$$

where

$$
\mathcal{X} = \left[\left(\eta^2 - 27 + 3\sqrt{81 - 6\eta^2} \right) \eta \right]^{1/3}.
$$

At $\eta = \eta^+ = 4$ the solutions $z_{5,6}$ collapse to the solution $z_2 = 0$ (see Fig. 1(b)).

Remarks.

– We emphasize that Eq. (38) admits, when $\sigma = 1$ (cubic nonlinearity) and $C_R = C_L$, an explicit solution by means of Jacobian elliptic function (see ref. 14 and the references therein).

– The qualitative behavior of the solutions of Eq. (38) could be easily studied by means of the conservation of the energy $\tilde{H}$ as done, for instance, by ref. 8.

6. STABILITY OF THE TWO-LEVEL APPROXIMATION

6.1. Main Result

Our main result consists in proving the stability of the two-level approximation. We prove that:

Theorem 5. Let $\psi_c = \Pi_c \psi$, $a_R = \langle \psi, \varphi_R \rangle$ and $a_L = \langle \psi, \varphi_L \rangle$, where ψ is the solution of Eq. (32), let b_R and b_L be the solution of the system of ordinary differential Eq. (36). Then, for any fixed $\tau' > 0$

$$|b_{R,L}(\tau) - a_{R,L}(\tau)| = \tilde{O}(e^{-\Gamma/\hbar}) \text{ and } \|\psi_c(\cdot, \tau)\| = \tilde{O}(e^{-\Gamma/\hbar}) \tag{40}$$

for any $\tau \in [0, \tau']$, where $\Gamma > 0$ is given in Eq. (16).

Remark.

– From this theorem, it follows that the time behavior, at least for times of the order of the beating period, of the wavefunction ψ, initially prepared on the lowest states, is practically described by means of the solutions of the two-level approximation given in the previous section.

6.2. Proof of Theorem 5

For the sake of simplicity, hereafter, let us drop out the parameters where this does not cause misunderstanding. The proof of the theorem is organized in several Lemmas.

Lemma 4. Let ψ be the solution of (32) and write according to (33)

$$\varphi(x, \tau) = a_R(\tau)\varphi_R(x) + a_L(\tau)\varphi_L(x) \text{ and } \psi = \psi_c + \varphi$$

Let $W\psi = W^I + W^{II}$ where:

(i) **Local perturbation:**

$$W^I = W_\ell^I = g(x)|\varphi(x, \tau)|^{2\sigma}\varphi(x, \tau), \tag{41}$$

where W_ℓ^I does not depend on ψ_c, and let

$$W_\ell^{II} = W_\ell \psi - W_\ell^I$$

Then, it follows that

$$\| W_\ell^I \| \leqslant C\hbar^{-\sigma d/2} \tag{42}$$

and

$$\| W_\ell^{II} \| \leqslant C\hbar^{-d\sigma/2} \| \psi_c \|, \tag{43}$$

for some positive constant C independent on τ and $\hbar$.

(ii) **Non-Local perturbation:**

$$W^I = W_{n\ell}^I = \langle \varphi(\cdot,\tau), g(\cdot)\varphi(\cdot,\tau)\rangle g(x)\varphi(x,\tau), \tag{44}$$

where $W_{n\ell}^I$ does not depend on ψ_c, and let

$$W_{n\ell}^{II} = W_{n\ell}\psi - W_{n\ell}^I.$$

Then, it follows that

$$\| W_{n\ell}^I \| \leqslant C \quad \text{and} \quad \| W_{n\ell}^{II} \| \leqslant C \| \psi_c \| \tag{45}$$

for some positive constant C independent on τ and $\hbar$.

Proof. The proof of this Lemma is given in several steps. At first we estimate the terms $W_{n\ell}^I$ and $W_{n\ell}^{II}$, then we give the proof of the estimates (42) and (43) for the local perturbation case; for what concerns the estimate of the term W_ℓ^{II} we consider different cases depending on the dimension d.

Proof of estimates (45). Let us consider, at first, the non-local case where $W_{n\ell}^I$ is given by (44), then

$$\begin{aligned} \| W_{n\ell}^I \| &\leqslant \tilde{g} |\langle \varphi(\cdot,\tau), g(\cdot)\varphi(\cdot,\tau)\rangle| \cdot \|\varphi\| \\ &\leqslant \tilde{g}^2 \|\varphi^2\|_1 \cdot \|\varphi\| \\ &\leqslant \tilde{g}^2 \|\varphi\|^3 \leqslant C, \end{aligned}$$

since $\|\varphi\| \leqslant \|\psi\| \leqslant 1$. For what concerns the other term it follows that

$$\begin{aligned} W_{n\ell}^{II} &= W_{n\ell}\psi - W_{n\ell}^I \\ &= g(x)[(\langle \psi_c, g\psi_c\rangle + \langle \varphi, g\psi_c\rangle + \langle \psi_c, g\varphi\rangle) g(\psi_c + \varphi) + \langle \varphi, g\varphi\rangle \psi_c]. \end{aligned}$$

Hence

$$\begin{aligned}\|W_{n\ell}^{II}\| &\leqslant C\left\{[\|\psi_c\|^2+2\|\varphi\|\cdot\|\psi_c\|](\|\psi_c\|+\|\varphi\|)+\|\varphi\|^2\|\psi_c\|\right\}\\ &\leqslant C\|\psi_c\|.\end{aligned}$$

Proof of estimate (42). The estimate of the term W_ℓ^I is immediate. Indeed

$$\|W_\ell^I\|\leqslant\tilde g\|\varphi^{2\sigma+1}\|=\tilde g\|\varphi\|_\infty^{2\sigma}\|\varphi\|\leqslant C\hbar^{-d\sigma/2}$$

since $\|\varphi\|\leqslant 1$ and (see Lemma 2)

$$\|\varphi\|_\infty\leqslant|a_R(\tau)|\cdot\|\varphi_R\|_\infty+|a_L(\tau)|\cdot\|\varphi_L\|_\infty\leqslant 2C\hbar^{-d/4}. \tag{46}$$

Proof of the estimate (43) – Dimension $d=1$. In such a case, the proof is simpler than the case of dimension 2, because we can make use of the inequality

$$\|\psi_c\|_\infty\leqslant C\hbar^{-d/4},\quad d=1, \tag{47}$$

which immediately follows in the case $d=1$ for any σ from the Minkowski inequality and from (31):

$$C\hbar^{-d\frac{p-2}{4p}}\geqslant\|\psi\|_p\geqslant-(|a_R(\tau)|\|\varphi_R\|_p+|a_L(\tau)|\|\varphi_L\|_p)+\|\psi_c\|_p,$$

for $p=+\infty$, where $|a_{R,L}(\tau)|\leqslant 1$ and where $\varphi_{R,L}$ satisfy the estimate (13). We consider

$$\begin{aligned}W_l^{II} &= W_l\psi-W_l^I=g(x)[|\psi_c+\varphi|^{2\sigma}(\psi_c+\varphi)-|\varphi|^{2\sigma}\varphi]\\ &= g(x)[|\psi_c+\varphi|^{2\sigma}-|\varphi|^{2\sigma}]\psi_c+g(x)[|\psi_c+\varphi|^{2\sigma}-|\varphi|^{2\sigma}]\varphi.\end{aligned} \tag{48}$$

From (48) and (A1) then it follows that

$$\|W_l^{II}\|\leqslant\tilde g[\|\varphi\|_\infty^{2\sigma}+\|\psi_c\|_\infty^{2\sigma}]\|\psi_c\|\leqslant C\hbar^{-\frac{1}{4}d(2\sigma)}\|\psi_c\|.$$

Proof of estimate (43) – Dimension $d=2$. In such a case, the estimate (47) does not hold and we make use here of the Gagliardo–Nirenberg inequality. Indeed, from (48) and (A1) it follows that

$$\|W_l^{II}\|\leqslant C[\|\varphi^{2\sigma}\psi_c\|+\|\psi_c^{2\sigma+1}\|]\leqslant C[\|\varphi\|_\infty^{2\sigma}\|\psi_c\|+\|\psi_c^{2\sigma+1}\|]. \tag{49}$$

The Gagliardo–Nirenberg inequality (A4) gives that

$$\|\psi_c^k\| = \|\psi_c\|_{2k}^k \leqslant C\|\nabla\psi_c\|^{k\delta}\|\psi_c\|^{(1-\delta)k},$$

where $k=2\sigma+1$ and $\delta=\frac{d}{2}-\frac{d}{2k}$. From this fact and from the estimate (31) it follows that

$$\|\psi_c^k\| \leqslant C\hbar^{-\frac{d}{4}(k-1)}\|\psi_c\|^{\frac{d}{2}+(1-\frac{d}{2})k} \leqslant C\hbar^{-d\frac{\sigma}{2}}\|\psi_c\| \tag{50}$$

since $d=2$ and $k=2\sigma+1$. ∎

Remarks.

– For what concerns the nonlinear local perturbation in dimension $d>2$ we emphasize that from Theorem 2 then $\psi=\psi_c+\varphi\in L^p$ for any $p<\frac{2d}{d-2}$ when $d>2$. Hence, in order to consider the L^2-norm of $W_l\psi$ we have to assume that $|\psi|^{2\sigma}\psi\in L^2$, that is $\sigma<\frac{1}{d-2}$. In such a case, then $W_l\psi\in L^2$ and we can apply the above estimate obtaining that

$$\|W_{l,a}^{II}\| \leqslant C\hbar^{-d\sigma/2}\|\psi_c\|^{\gamma},\ \gamma=1+(2-d)\sigma$$

since $\|\psi_c\|\leqslant 1$, and, similarly,

$$\|W_{l,b}^{II}\| \leqslant C\hbar^{-d\sigma/2}\|\psi_c\|^{\gamma}$$

Lemma 5. $|a'_{R,L}|\leqslant C$ for any $\tau\geqslant 0$ for some positive constant C independent of τ and $\hbar$.

Proof. In the local perturbation case from (34) and from the previous Lemma it follows that

$$\begin{aligned}|a'_R| \leqslant\ & |a_L|+|r_R| \leqslant |a_L|+\frac{|\epsilon|}{\omega}\|\varphi_R\|\cdot\|W\psi\| \\ \leqslant\ & |a_L|+\frac{|\epsilon|}{\omega}\|\varphi_R\|[\|W_l^I\|+\|W_l^{II}\|] \\ \leqslant\ & |a_L|+C\frac{|\epsilon|\hbar^{-\sigma d/2}}{\omega}\|\varphi_R\|[1+\|\psi_c\|]\leqslant C,\end{aligned}$$

since $|a_L|\leqslant 1$ and $\|\psi_c\|\leqslant 1$ for any τ, $\|\varphi_R\|=1$ and (22). In the same way, the estimate $|a'_L|\leqslant C$ follows. The non-local perturbation case similarly follows. ∎

Lemma 6. Let $W_l^I = W_l^I(x,\tau)$ and $W_{nl}^I = W_{nl}^I(x,\tau)$ be defined as in (41) and (44); then

$$W_l^I, W_{nl}^I \in C^1(\mathbb{R}, L^2(\mathbb{R}^d))$$

and

$$\left\| \frac{\partial W_l^I}{\partial \tau} \right\|, \left\| \frac{\partial W_{nl}^I}{\partial \tau} \right\| \leqslant C\hbar^{-\sigma d/2}, \quad \forall \tau \geqslant 0 \quad \text{and} \quad \forall \hbar \in (0, \hbar^*).$$

Proof. Let us consider, for a moment, the local case where

$$\begin{aligned} W_l^I &= W_l^I(x,\tau) \\ &= g(x)|a_R(\tau)\varphi_R(x) + a_L(\tau)\varphi_L(x)|^{2\sigma}(a_R(\tau)\varphi_R(x) + a_L(\tau)\varphi_L(x)) \\ &= g(x)[\bar{a}_R(\tau)\bar{\varphi}_R(x) + \bar{a}_L(\tau)\bar{\varphi}_L(x)]^{\sigma}[a_R(\tau)\varphi_R(x) + a_L(\tau)\varphi_L(x)]^{\sigma+1}. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial W_l^I}{\partial \tau} = g(x) \Big\{ & \sigma(\bar{a}_R\bar{\varphi}_R + \bar{a}_L\bar{\varphi}_L)^{\sigma-1}(a_R\varphi_R + a_L\varphi_L)^{\sigma+1}(\bar{a}'_R\bar{\varphi}_R + \bar{a}'_L\bar{\varphi}_L) \\ & + (\sigma+1)|\bar{a}_R\bar{\varphi}_R + \bar{a}_L\bar{\varphi}_L|^{2\sigma}(a'_R\varphi_R + a'_L\varphi_L) \Big\}. \end{aligned}$$

Hence,

$$\left\| \frac{\partial W_l^I}{\partial \tau} \right\| \leqslant \bar{g}(2\sigma+1)\max[|a'_R|, |a'_L|]\max[\|\varphi_R\|_\infty^{2\sigma}, \|\varphi_L\|_\infty^{2\sigma}] \leqslant C\hbar^{-\sigma d/2},$$

where $|a'_{R,L}| \leqslant C$ from the previous Lemma. A similar estimate for the non-local perturbation case follows; where we can differentiate with respect to τ under the integral $\langle \varphi, g\varphi \rangle$ since the integral converges uniformly with respect to τ. ▮

We give now an *a priori* estimate of the term ψ_c.

Lemma 7. Let $\psi_c = \Pi_c \psi$ where ψ is the solution of Eq. (32); it satisfies to the following estimate

$$e^{-C\tau}\|\psi_c\| = \tilde{O}(e^{-\Gamma/\hbar}), \quad \forall \tau \geqslant 0 \tag{51}$$

for some positive constant $C > 0$ independent of $\hbar$ and τ.

Remarks.

– In particular, from (51) it follows that

$$\|\psi_c\| = \tilde{O}(e^{-\Gamma/\hbar})$$

uniformly for any $\tau \in [0, \tau']$ for any τ' fixed, from which the second estimate (40) follows.

Proof. Now, in order to prove the estimate (51) we first deduce from (34) the relation

$$\begin{aligned} \psi_c(\cdot, \tau) &= -i \int_0^\tau e^{-i(H_0-\Omega)(\tau-s)/\omega} r_c \, ds \\ &= -i \frac{\epsilon}{\omega} \int_0^\tau e^{-i(H_0-\Omega)(\tau-s)/\omega} \Pi_c W(\cdot, \psi)\psi(\cdot, s) ds, \end{aligned} \tag{52}$$

since $\psi_c^0 = \Pi_c \psi^0 = 0$ from assumption Hypothesis 2. Therefore, we can write

$$\psi_c = -i \frac{\epsilon}{\omega} [I + II]$$

where

$$\begin{aligned} I &= \int_0^\tau e^{-i(H_0-\Omega)(\tau-s)/\omega} \Pi_c W^I \mathrm{ds}, \quad W^I = W^I(\varphi) \\ II &= \int_0^\tau e^{-i(H_0-\Omega)(\tau-s)/\omega} \Pi_c W^{II} \mathrm{ds}, \quad W^{II} = W^{II}(\varphi, \psi_c). \end{aligned}$$

For what concerns the first term it follows that, by integrating by part,

$$\begin{aligned} I = &\left[-i\omega e^{-i(H_0-\Omega)(\tau-s)/\omega} [H_o - \Omega]^{-1} \Pi_c W^I \right]_0^\tau \\ &+ i\omega \int_0^\tau e^{-i(H_0-\Omega)(\tau-s)/\omega} [H_0 - \Omega]^{-1} \Pi_c \frac{\partial W^I}{\partial s} ds. \end{aligned}$$

Let us emphasize that

$$\|e^{-i(H_0-\Omega)(\tau-s)/\omega}\| = 1$$

and from Lemma 1 it follows that

$$\|\hbar[H_0 - \Omega]^{-1} \Pi_c\| \leqslant C.$$

From these facts and from Lemmas 4 and 6 then

$$\|I\| \leqslant C\frac{\omega}{\hbar}\max_{s\in[0,\tau]}\left\{\|W^I\| + \tau\left\|\frac{\partial W^I}{\partial s}\right\|\right\} \leqslant C\frac{\omega}{\hbar}\hbar^{-\sigma d/2}(1+\tau).$$

For what concerns the other term it follows that

$$\|II\| \leqslant \int_0^\tau \|W^{II}\|\mathrm{ds} \leqslant C\hbar^{-\sigma d/2}\int_0^\tau \|\psi_c\|\mathrm{ds}.$$

Collecting all these results and denoting

$$h(\tau) = \|\psi_c(\cdot,\tau)\|,$$

then $h(\tau)$ is a non-negative real-valued function satisfying the estimate

$$\begin{aligned} h(\tau) &\leqslant \frac{\epsilon}{\omega}\left\{C\omega\hbar^{-1-\sigma d/2}(1+\tau) + C\hbar^{-\sigma d/2}\int_0^\tau h(s)d\mathrm{s}\right\} \\ &\leqslant a\int_0^\tau h(s)ds + b(1+\tau), \end{aligned} \tag{53}$$

where

$$a = C\frac{\epsilon\hbar^{-\sigma d/2}}{\omega} = C\eta = O(1)$$

since (22) and

$$b = C\epsilon\hbar^{-1-\sigma d/2} = \tilde{O}\left(e^{-\Gamma/\hbar}\right)$$

since (22) and (17). Then, the Gronwall's Lemma gives that

$$h(\tau) \leqslant be^{a\tau} + \frac{b}{a}[e^{a\tau} - 1] \leqslant Cbe^{C\tau}$$

proving so the estimate (51). ∎

Remark.

– In dimension $d>2$ for local perturbation case and where $\sigma<\frac{1}{d-2}$, then (53) takes the form

$$h(\tau)\leqslant a\int_0^\tau h^\gamma(s)\mathrm{d}s+b(1+\tau), \quad \gamma=1+(2-d)\sigma\in(0,1),$$

from which and by means of Gronwall's Lemma arguments the *a priori* weaker estimate follows

$$h(\tau)\leqslant[b+a\tau]^{1/(1-\gamma)}.$$

Unfortunately, this estimate is not useful in order to extend the result of Lemma 7 to the case of dimension $d>2$.

Now, we are ready to complete the proof of the theorem. Let

$$J=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, \quad A=\begin{pmatrix}a_R\\ a_L\end{pmatrix}, \quad R=\begin{pmatrix}-ir_R\\ -ir_L\end{pmatrix},$$

then the system (34) takes the form

$$\begin{cases}A'=JA+R\\ \psi_c'=-\frac{i}{\omega}[H_0-\Omega]\psi_c-ir_c, \quad R=R(A,\psi_c).\end{cases} \tag{54}$$

Let

$$B=\begin{pmatrix}b_R\\ b_L\end{pmatrix}, \quad \tilde{R}=\begin{pmatrix}-i\eta\tilde{r}_R\\ -i\eta\tilde{r}_L\end{pmatrix},$$

then the *two-level approximation* (36) takes the form

$$B'=JB+\tilde{R}, \quad \tilde{R}=\tilde{R}(B). \tag{55}$$

We emphasize that

$$A,B\in S^2,$$

where

$$S^2=\left\{A=\begin{pmatrix}a_R\\ a_L\end{pmatrix}, a_R, a_L\in\mathbb{C}:|A|=\sqrt{|a_R|^2+|a_L|^2}\leqslant 1\right\}.$$

Let now

$$F : S^2 \to C^2$$

defined as

$$F(A) = JA + \tilde{R}(A). \tag{56}$$

Hence, the first Eq. of (54) and Eq. (55) can be written as

$$B' = F(B) \quad \text{and} \quad A' = F(A) + \left[R(A, \psi_c) - \tilde{R}(A) \right]. \tag{57}$$

Lemma 8. The function F defined in Eq. (56) satisfies to the following Lipschitz condition

$$|F(A) - F(B) \leqslant C|A - B| \tag{58}$$

for some $C > 0$.

Proof. By Definition

$$F(A) - F(B) = J(A - B) + [\tilde{R}(A) - \tilde{R}(B)],$$

where

$$\tilde{R}(A) - \tilde{R}(B) = \begin{pmatrix} \eta[\tilde{r}_R(A) - \tilde{r}_R(B)] \\ \eta[\tilde{r}_L(A) - \tilde{r}_L(B)] \end{pmatrix}.$$

In the local perturbation case, it follows that (we assume, for definiteness, that $|b_R| \leqslant |a_R|$)

$$\begin{aligned} \tilde{r}_R(A) - \tilde{r}_R(B) &= C_R[|a_R|^{2\sigma} a_R - |b_R|^{2\sigma} b_R] \\ &= C_R[|a_R|^{2\sigma}(a_R - b_R) + (|a_R|^{2\sigma} - |b_R|^{2\sigma}) b_R] \\ &= C_R[|a_R|^{2\sigma}(a_R - b_R) + (|a_R|^{\sigma} - |b_R|^{\sigma})(|a_R|^{\sigma} + |b_R|^{\sigma}) b_R], \end{aligned}$$

hence

$$|\tilde{r}_R(A) - \tilde{r}_R(B)| \leqslant C|a_R - b_R|$$

since (A1) and since $|a_R|, |b_R| \leqslant 1$. The estimate for the term $\tilde{r}_L(A) - \tilde{r}_L(B)$ similarly follows. In the non-local case it follows that

$$\begin{aligned}\tilde{r}_R(A) - \tilde{r}_R(B) &= C_R[|a_R|^2 a_R - |b_R|^2 b_R] + \tilde{C}[a_R|a_L|^2 - b_R|b_L|^2] \\ &= C_R[|a_R|^2(a_R - b_R) + (|a_R|^2 - |b_R|^2)b_R] \\ &\quad + \tilde{C}[(a_R - b_R)|a_L|^2 + b_R(|a_L|^2 - |b_L|^2)]\end{aligned}$$

from which follows (58) immediately, similarly for $r_L(A) - r_L(B)$. ∎

From (57) it follows that

$$(A-B)' = F(A) - F(B) + [R(A, \psi_c) - \tilde{R}(A)], \text{ where } A(0) = B(0). \quad (59)$$

Lemma 9. For any $\Gamma', 0 < \Gamma' < \Gamma$ there exist $\hbar^\star > 0$ and $C > 0$, independent of $\hbar$ and τ, such that

$$|R(A, \psi_c) - \tilde{R}(A)| \leqslant Ce^{-\Gamma'/\hbar} e^{C\tau}, \quad \forall \tau \geqslant 0, \quad \forall \hbar \in (0, \hbar^\star).$$

Proof. Indeed (for the sake of definiteness we consider only one term),

$$\begin{aligned}&r_R(a_R, a_L, \psi_c) - \eta \tilde{r}_R(a_R, a_L) \\ &\quad = [r_R(a_R, a_L, \psi_c) - r_R(a_R, a_L, 0)] + [r_R(a_R, a_L, 0) - \eta \tilde{r}_R(a_R, a_L)] \\ &\quad = [r_R(a_R, a_L, \psi_c) - r_R(a_R, a_L, 0)] + \tilde{O}(e^{-\Gamma/\hbar})\end{aligned}$$

since (35). Moreover,

$$r_R(a_R, a_L, \psi_c) = \frac{\epsilon}{\omega} \langle \varphi_R, W\psi \rangle,$$

where $W\psi = W^I + W^{II}$ and where W^I does not depend on ψ_c. From this fact and from Lemmas 4 and 7 it follows that

$$\begin{aligned}|r_R(a_R, a_L, \psi_c) - r_R(a_R, a_L, 0)| &= \frac{|\epsilon|}{\omega} |\langle \varphi_R, W^{II} \rangle| \\ &\leqslant \frac{|\epsilon|}{\omega} \|\varphi_R\| \cdot \|W^{II}\| \leqslant \frac{|\epsilon|}{\omega} C\hbar^{-d\sigma/2} \|\psi_c\| \\ &\leqslant Ce^{-\Gamma'/\hbar} e^{C\tau}\end{aligned}$$

for any $\Gamma' \in (0, \Gamma)$ since (51). ∎

Now, we are ready to complete the proof of the theorem. From (59) it follows that

$$A(\tau)-B(\tau)=\int_0^\tau [F[A(s)]-F[B(s)]]ds+\int_0^\tau [R(A,\psi_c)-\tilde{R}(A)]ds.$$

If we set

$$q(\tau)=|A(\tau)-B(\tau)|$$

then from this equation and from Lemmas 8 and 9 it follows that

$$q(\tau)\leqslant C\int_0^\tau q(s)\mathrm{d}s+Ce^{-\Gamma'/\hbar}[e^{C\tau}-1]$$

for any $\Gamma'\in(0,\Gamma)$, from which and from Gronwall's Lemma the desired estimate (40) follows. The proof of the Theorem is so completed.

7. APPENDIX A: INEQUALITIES

Basic Inequality. Let $\gamma>0$. Then, for any $y\geqslant x>0$, it follows that

$$[y^\gamma - x^\gamma]\leqslant \gamma(y-x)[y^{\gamma-1}+x^{\gamma-1}]. \tag{A1}$$

Indeed, the Taylor expansion, up to the first term, with a remainder term gives that

$$y^\gamma = x^\gamma+\gamma(y-x)\bar{x}^{\gamma-1}$$

for some $\bar{x}\in(x,y)$; from which the estimate (A1) immediately follows.

Gagliardo–Nirenberg inequality. The Gagliardo–Nirenberg inequality states that

$$\|f\|_{2\sigma+2}^{2\sigma+2}\leqslant C_{\sigma,d}\|\nabla f\|_2^{\sigma d}\|f\|_2^{2+\sigma(2-d)}, \tag{A2}$$

where

$$\sigma\in\begin{cases}[0,+\infty] & \text{if } d=1\\ [0,+\infty) & \text{if } d=2\\ [0,2/(d-2)) & \text{if } d>2\end{cases} \tag{A3}$$

and where C is a given constant. Such an inequality (A2) can be also rewritten as

$$\|f\|_p \leqslant C_{p,d}\|\nabla f\|^{\delta}\|f\|^{1-\delta}, \quad \delta = \frac{\sigma d}{2(\sigma+1)} = \frac{(p-2)d}{2p} = \frac{d}{2} - \frac{d}{p}, \tag{A4}$$

where

$$p \in \begin{cases} [2, +\infty] & \text{if } d = 1 \\ [2, +\infty) & \text{if } d = 2 \\ [2, 2d/(d-2)) & \text{if } d > 2. \end{cases}$$

ACKNOWLEDGMENTS

This work is partially supported by the Italian MURST and INDAM-GNFM (project *Comportamenti Classici in Sistemi Quantistici*).

REFERENCES

1. W. H. Aschbacher, J. Froehlich, G. M. Graf, K. Schnee, and M. Troyer, Symmetry breaking regime in the nonlinear Hartree equation, *J. Math. Phys.* **43**:3879–3891 (2002).
2. F. A. Berezin and M. A. Shubin, The Schrödinger equation, (Kluwer Academic publishers, 1991).
3. R. Carles, Nonlinear Schrödinger equations with repulsive harmonic potential and applications, *SIAM J. Math. Anal.* **35**:823–843 (2003).
4. T. Cazenave and F. B. Weissler, The Cauchy problem for the nonlinear Schrödinger equation in H^1, *Manuscripta Math.* **61**:477–494 (1988).
5. T. Cazenave, Semilinear Schrödinger Equations: *Courant Lecture Notes in Mathematics* (New York, 2003).
6. J. Ginibre, An Introduction to Nonlinear Schrödinger Equations *in Nonlinear waves*, R. Agemi, Y. Giga, T. Ozawa, eds. (1997).
7. V. Grecchi and A. Martinez Non-linear Stark effect and molecular localization, *Comm. Math. Phys.* **166**:533–548 (1995).
8. V. Grecchi, A. Martinez and A. Sacchetti, Destruction of the beating effect for a non-linear Schrödinger equation, *Comm. Math. Phys.* **227**:191–209 (2002).
9. B. Helffer and J. Sjöstrand, Multiple wells in the semi-classical limit I. *Comm. P.D.E.* **9**:337–408 (1984).
10. B. Helffer, Semi-Classical Analysis for the Schrödinger Operator and Applications: *Lecture Notes in Mathematics* 1336 (Springer-Verlag: Berlin Heidelberg, 1988).
11. G. Jona-Lasinio, C. Presilla, and C. Toninelli, Interaction induced localization in a gas of pyramidal molecules, *Phys. Rev. Lett.* **88**:123001 (2002).
12. E. H. Lieb, R. Seiringer, J. Yngvason, A rigorous derivation of the Gross–Pitaevskii energy functional for a two-dimensional Bose gas, *Comm. Math. Phys.* **224**:17–31 (2001).
13. S. Raghavan, A. Smerzi, S. Fantoni, and S. R. Shenoy Coherent oscillations between two weakly coupled Bose–Einstein condensates: Josephson effects, π oscillations, and macroscopic quantum self-trapping, *Phys. Rev. A* **59**:620–633 (1999).

14. A. Sacchetti, Nonlinear time-dependent one-dimensional Schrödinger equation with double well potential, *SIAM J. Math. Anal.* **35**:1160–1176 (2003).
15. C. Sulem and P. L. Sulem, The Nonlinear Schrödinger Equation: Self-focusing and Wave Collapse. (Springer-Verlag, New York 1999).
16. A. Vardi and J. R. Anglin, Bose–Einstein condensates beyond mean field theory: Quantum back reaction as decoherence, *Phys. Rev. Lett.* **86**:568–571 (2001).